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A Law of Large Numbers and a Central Limit Theorem for the Schrödinger Operator with Zero-Range Potentials

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We consider the Schrödinger operator with zero-range potentials on N points of three-dimensional space, independently chosen according to a common distribution $V(x)$. Under some assumptions we prove that, when N goes to infinity, the sequence converges to a Schrödinger operator with an effective potential. The fluctuations around the limit operator are explicitly characterized.

KEY WORDS: Zero-range potentials; law of large numbers; effective potentials.

1. INTRODUCTION

The study of the Schrödinger operator with zero-range interactions in three dimensions goes back to 1935, when Bethe and Peierls[1] used this model to study the system consisting of a neutron and a proton. Fermi also employed this model when analyzing scattering of neutrons in hydrogenous substances (e.g., parafin) in 1936[2] (see also Ref. 3). Thomas[4] showed how to obtain the Hamiltonian with zero-range interactions (point interactions) from Hamiltonians with suitably scaled short-range interaction. The rigorous analysis was initiated in 1961 by Berezin and Faddeev[5] in an attempt to treat the three-body problem. Their technique was to study certain self-adjoint extensions of the Laplacian suitably restricted. By now there are several ways to define rigorously a Schrödinger operator in three

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dimensions with finitely or infinitely many point interactions located on a discrete set in $\mathbf{R}^3$. For a comprehensive mathematical treatment of Schrödinger operators with point interactions in one to three dimensions we refer to Ref. 6. The easiest way mathematically to introduce the Schrödinger operator $-\Delta_{\alpha, Y}$ with point interactions located on the set $Y = \{y_1, ..., y_N\}$ with strength $\alpha = (\alpha_1, ..., \alpha_N)$ is to define it as the unique self-adjoint operator $-\Delta_{\alpha, Y}$ in $L^2(\mathbf{R}^3)$ whose resolvent equals

$$(-\Delta_{\alpha, Y} - k^2)^{-1} = G_k + \sum_{j, j'=1}^{N} [\Gamma_{\alpha, Y}(k)^{-1}]_{jj'} \overline{G_k(\cdot - y_{j'})}\, G_k(\cdot - y_j) \quad (1.1)$$

where $\Gamma_{\alpha, Y}(k)$ is the $N \times N$ matrix

$$\Gamma_{\alpha, Y}(k) = \left[\left(\alpha_j - \frac{ik}{4\pi}\right) \delta_{jj'} - \tilde{G}_k(y_j - y_{j'})\right]_{j, j'=1}^{N}, \qquad \operatorname{Im} k > 0 \quad (1.2)$$

and

$$\begin{aligned} G_k &= (-\Delta - k^2)^{-1}, \qquad \operatorname{Im} k > 0 \\ \tilde{G}_k(x) &= \begin{cases} G_k(x), & x \neq 0 \\ 0, & x = 0 \end{cases} \\ G_k(x) &= \frac{e^{ik|x|}}{4\pi |x|} \end{aligned} \quad (1.3)$$

Let us comment briefly on the definition given above.

For

$$\frac{|k|}{\inf_j \alpha_j} \gg 1 \qquad \text{and} \qquad (\inf_j \alpha_j) \inf_{j' \neq j} |y_j - y_{j'}| \gg 1$$

we get $[\Gamma_{\alpha, Y}(k)]_{jj'} \sim \alpha_j \delta_{jj'}$. In this approximation

$$(-\Delta_{\alpha, Y} - k^2)^{-1}(x, y) = G_k(x - y) + \sum_{i=1}^{N} \frac{1}{\alpha_i} \overline{G_k(x - y_i)}\, G_k(y - y_i) \quad (1.4)$$

which formally can be written

$$(-\Delta_{\alpha, Y} - k^2)^{-1} = (-\Delta - k^2)^{-1} - (-\Delta - k^2)^{-1} W (-\Delta - k^2)^{-1} \quad (1.5)$$

with

$$W(x) = -\sum_{i=1}^{N} \frac{1}{\alpha_i} \delta(x - y_i)$$

Then (1.4) is what one obtains by expanding the formal operator

$$\left[-\Delta-\sum_{i=1}^{N}\frac{1}{\alpha_i}\delta(x-y_i)-k^2\right]^{-1}$$

up to the first order in a Neumann series. Equation (1.4) has been the basic formula for the computation of scattering data in important applications, for example, neutron scattering by liquids and solids. See, e.g., Ref. 7.

As an aside, we mention that the formal expansion (1.5) cannot be continued beyond the first term without giving rise to divergences. On the contrary, from (1.1) one can get a low-energy, small-scattering-length expansion which is free of divergences. To the best of our knowledge this possibility has yet to be exploited in applications.

Let V be any positive density distribution such that

$$V(x)\geqslant 0, \qquad \int_{\mathbf{R}^3} V(x)\,dx=1, \qquad \|V\|_2<\infty$$

In particular, V belongs to the Rollnik class, i.e.,

$$\|V\|_R^2\equiv\int_{\mathbf{R}^3}\frac{|V(x)|\,|V(y)|}{|x-y|^2}\,dx\,dy\leqslant c\,\|V\|_1^{1/3}\,\|V\|_2^{2/3}<\infty$$

(see, for example, Ref. 8).

On the set of configurations $Y^{(N)}=\{y_1^{(N)},\ldots,y_N^{(N)}\}$ of N points of $\mathbf{R}^3$ we consider the probability measure $\{V(x)\,dx\}^{\otimes N}$ [in such a way that the points $y_i^{(N)}$ are identically and independently distributed according to the common density $V(x)$].

Given any real function $\alpha(x)$, $x\in\mathbf{R}^3$, such that $0<C_1<|\alpha|\,(x)<c_2<\infty$, continuous apart from a set of $V(x)\,dx$-measure zero, we define $\alpha^{(N)}\equiv\{\alpha(y_1^{(N)}),\ldots,\alpha(y_N^{(N)})\}$.

By the law of large numbers we have

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{i=1}^{N}\overline{G_k(x-y_i^{(N)})}\,\frac{1}{\alpha(y_i^{(N)})}\,G_k(y_i^{(N)}-y)$$

$$=\int_{\mathbf{R}^3}\overline{G_k(x-z)}\,\frac{V(z)}{\alpha(z)}\,G_k(z-y)\,dz$$

$$=\left[(-\Delta-k^2)^{-1}\frac{V}{\alpha}(-\Delta-k^2)^{-1}\right](x,y)$$

At least at the formal level expressed by (1.4) and (1.5) it is then suggested that

$$(-\Delta_{N\alpha^{(N)},Y^{(N)}}-k^2)^{-1}\xrightarrow[N\uparrow\infty]{}\left(-\Delta-\frac{V}{\alpha}-k^2\right)^{-1}$$

Notice that any potential $U(x) \in L_2 \cap L_1$ can be written as the ratio V/α of a density distribution in L_2 with a function α satisfying the assumptions stated above. In fact, it is enough to define

$$V = |U|\,(x) \Big/ \int_{\mathbf{R}^3} |U|\,(x)\,dx$$

$$\alpha = (\text{sign } U)(x) \Big/ \int_{\mathbf{R}^3} |U|\,(x)\,dx$$

The aim of this paper is to give a proof of a law of large numbers for Hamiltonians with point interactions, thereby proving that any Schrödinger operator with potential in $L_2 \cap L_1$ is a limit of a sequence of such Hamiltonians. Moreover, the fluctuations around the limit operator can be completely characterized.

Our main result will be:

Theorem 1. Under the assumptions on V, $Y^{(N)}$, and $\alpha^{(N)}$ made above, uniformly on a set of configurations $Y^{(N)}$ of measure increasing to 1 as N goes to infinity, for $k = i\sqrt{\lambda}$, λ positive large enough,

$$\underset{N\uparrow\infty}{s\text{-lim}}\,(-\Delta_{N\alpha^{(N)},\,Y^{(N)}} + \lambda)^{-1} = \left(-\Delta - \frac{V}{\alpha} + \lambda\right)^{-1} \equiv A_V^\lambda \tag{1.7}$$

In the next section we will give a proof of Theorem 1 based on an analogous result proved in Ref. 9.

The result on the fluctuations around the limit operator A_V^λ is expressed in the following:

Theorem 2. For any f, $g \in L_2(\mathbf{R}^3)$ the random variable

$$N^{1/2}(f, [(-\Delta_{N\alpha^{(N)},\,Y^{(N)}} + \lambda)^{-1} - A_V^\lambda]\, g) \equiv \xi_{f,g}^N(Y^{(N)})$$

converges in distribution, when N goes to ∞, to the Gaussian random variable $\xi_{f,g}$ of mean zero and variance

$$E(\xi_{f,g}^2) = \left(A_V^\lambda f A_V^\lambda g, \frac{1}{\alpha^2} A_V^\lambda f A_V^\lambda g\right)_{L_V^2} - \left(A_V^\lambda f, \frac{1}{\alpha} A_V^\lambda g\right)_{L_V^2}^2$$

where

$$(s, s')_{L_V^2} = \int_{\mathbf{R}^3} s(x)\, s'(x)\, V(x)\, dx$$

Only slight modifications of the steps followed in Ref. 10 for a similar problem are required to obtain a proof of Theorem 2. Rather than reproduce the proof here, we refer the reader to that paper.

2. THE LAW OF LARGE NUMBERS FOR HAMILTONIANS WITH POINT INTERACTIONS

In order to stress the analogy with the problem analyzed in Refs. 9 and 10, which we will follow closely, we briefly mention the relation between operator (1.1) and the "approximate" Green's function of a boundary value problem on spheres shrinking to the points of Y.

It is outside the scope of this paper to state rigorously the connection between the two problems. We are convinced, however, that the asymptotics of the associated boundary value problem for a finite or an infinite number of points could be important in applications. We plan to come back to this in further work.

Let B_i^r be the closed sphere of radius r around $y_i \in Y$, $B_i^r = \{x \in \mathbf{R}^3 \mid |x - y_i| \leqslant r\}$. Consider the problem

$$\begin{aligned} &-(\Delta u)(x) + \lambda u(x) = f(x), && x \in R^3 \Big\backslash \bigcup_{i=1}^{N} B_i^r \\ &u(x) + \gamma_i(r) \frac{\partial u}{\partial n}(x) = 0, && x \in \partial B_i^r, \quad i = 1,..., N \end{aligned} \tag{2.1}$$

where $\gamma_i(r)$ are N bounded functions of $r \in [0, 1]$ and $\partial/\partial n$ indicates normal derivative in the direction outgoing from the spheres.

As for the Dirichlet problem in Refs. 9 and 10, for small r, we will try to approximate the effects of surface terms with point sources placed at the points of Y in the form

$$\tilde{u}(x) = (G_{i\sqrt{\lambda}} f)(x) + \sum_{j=1}^{N} q_f^j G_{i\sqrt{\lambda}}(x - y_j)$$

By direct computation [we will assume enough regularity on f to guarantee existence and boundedness of $(\partial/\partial n)(G_{i\sqrt{\lambda}} f)$], for $x \in \partial B_i^r$

$$\begin{aligned} &\tilde{u}(x) + \gamma_i(r) \frac{\partial \tilde{u}}{\partial n}(x) \\ &\quad = (G_{i\sqrt{\lambda}} f)(y_i) + [(G_{i\sqrt{\lambda}} f)(x) - (G_{i\sqrt{\lambda}} f)(y_i)] \\ &\qquad + \gamma_i(r) \left(\frac{\partial}{\partial n} (G_{i\sqrt{\lambda}} f) \right)(x) + \sum_{j=1}^{N} q_f^j \tilde{G}_{i\sqrt{\lambda}}(y_i - y_j) \\ &\qquad + \sum_{j=i}^{N} q_f^j [\tilde{G}_{i\sqrt{\lambda}}(x - y_j) - \tilde{G}_{i\sqrt{\lambda}}(y_i - y_j)] \\ &\qquad + q_f^i \frac{\exp(-\lambda^{1/2} r)}{4\pi r} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \gamma_i(r)\, q_f^j \frac{\partial G_{i\sqrt{\lambda}}}{\partial n}(x - y_j) \\ &\qquad - \gamma_i(r)\, q_f^i \lambda^{1/2} \frac{\exp(-\lambda^{1/2} r)}{4\pi r} - \gamma_i(r)\, q_f^i \frac{\exp(-\lambda^{1/2} r)}{4\pi r^2}, \qquad x \in \partial B_i^r \end{aligned}$$

To cancel singular terms, one has to choose $\gamma_i(r) = r + \alpha_i \cdot 4\pi r^2$. Furthermore, if now the charges q_f^i are chosen in such a way that they satisfy the linear system of equations

$$\sum_{j=1}^{N} \left[\left(\alpha_j + \frac{\sqrt{\lambda}}{4\pi}\right)\delta_{ij} - \tilde{G}_{i\sqrt{\lambda}}(y_i - y_j)\right] q_f^j = (G_{i\sqrt{\lambda}} f)(y_i)$$

or

$$q_f^i = \sum_{j=1}^{N} [\Gamma_{\alpha, Y}(i\sqrt{\lambda})^{-1}]_{ij} (G_{i\sqrt{\lambda}} f)(y_j)$$

one gets

$$\tilde{u}(x) + \gamma_i(r) \frac{\partial \tilde{u}}{\partial n}(x) = O(r), \qquad x \in \partial B_i^r, \quad i = 1,..., N$$

In this sense, point interactions located at the points of Y act like mixed boundary conditions at each of the points of Y.

We now turn to the proof of Theorem 1. From the definition, for any $f, g \in L_2(\mathbf{R}^3)$,

$$\begin{aligned}(f, (-\Delta_{N\alpha^{(N)}, Y^{(N)}} + \lambda)^{-1} g) &= (f, G_{i\sqrt{\lambda}} g) + \sum_{i,j=1}^{N} [\Gamma^{-1}_{N\alpha^{(N)}, Y^{(N)}}(i\sqrt{\lambda})]_{ij} \\ &\quad \times (G_{i\sqrt{\lambda}} f)(y_j^{(N)})(G_{i\sqrt{\lambda}} g)(y_i^{(N)}) \qquad (2.2)\end{aligned}$$

The above discussion suggests splitting (2.2) into

$$(f, (-\Delta_{N\alpha^{(N)}, Y^{(N)}} + \lambda)^{-1} g) = (f, G_{i\sqrt{\lambda}} g) + \sum_{i=1}^{N} q^i_{\lambda, g} (G_{i\sqrt{\lambda}} f)(y_i^{(N)}) \qquad (2.3)$$

where

$$\begin{aligned}&\left(N\alpha\,(y_j^{(N)}) + \frac{\sqrt{\lambda}}{4\pi}\right) q^j_{\lambda, g} - \sum_{i=1}^{N} \tilde{G}_{i\sqrt{\lambda}}(y_i^{(N)} - y_j^{(N)})\, q^i_{\lambda, g} \\ &\quad = (G_{i\sqrt{\lambda}} g)(y_j^{(N)}), \qquad j = 1,..., N \qquad (2.4)\end{aligned}$$

To show that the linear system (2.4) is solvable, in such a way that it can be taken as an unambiguous definition of the $q_{\lambda, g}$, we notice first that

$$\left\| \frac{\tilde{G}_{i\sqrt{\lambda}}}{N} \right\|_{\text{H.S.}} = \left\{ \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\exp(-2\lambda^{1/2}\, |y_i^{(N)} - y_j^{(N)}|)}{16\pi^2\, |y_i^{(N)} - y_j^{(N)}|^2} \right\}^{1/2} \qquad (2.5)$$

where $\|A\|_{\mathrm{H.S.}}$ denotes the Hilbert–Schmidt norm of a matrix A, and where

$$(\tilde{G}_{i\sqrt{\lambda}})_{ij} = \tilde{G}_{i\sqrt{\lambda}}(y_i^{(N)} - y_j^{(N)})$$

By the law of large numbers and by our assumptions on the density distribution V, we then have

$$\left\| \frac{\tilde{G}_{i\sqrt{\lambda}}}{N} \right\|_{\mathrm{H.S.}} \leqslant c(\lambda)\, \|V\|_R$$

with $\lim_{\lambda\uparrow\infty} c(\lambda) = 0$, $\|V\|_R < \infty$.

This implies that the matrix

$$\frac{1}{N} \Gamma_{N\alpha^{(N)},\, Y^{(N)}}(i\sqrt{\lambda})$$

[where $\alpha_i \equiv \alpha(y_i^{(N)})$] has a bounded inverse for an appropriate choice of λ and N sufficiently large.

In order to analyze the properties of the $q^i_{\lambda,g}$ when N is very large, we consider the integral equations corresponding to (2.4) in the continuum

$$\alpha(x)\, q_{\lambda,g}(x) - \int_{\mathbf{R}^3} G_{i\sqrt{\lambda}}(x-y)\, V(y)\, q_{\lambda,g}(y)\, dy = (G_{i\sqrt{\lambda}}\, g)(x) \qquad (2.6)$$

which is solved, for λ large enough, by

$$\alpha(x)\, q_{\lambda,g}(x) = [(-\Delta - V/\alpha + \lambda)^{-1} g](x)$$

Notice that, by our assumptions on V, such a function belongs to the operator domain of the Laplacian. In particular, it is continuous, implying the continuity, $V(x)\,dx$-almost everywhere of the function $q_{\lambda,g}(x)$.

Comparing (2.6) with (2.4), we find

$$\sum_{j=1}^{N} \left[\left(\alpha_i + \frac{\sqrt{\lambda}}{4\pi N} \right) \delta_{ij} - \left(\frac{\tilde{G}_{i\sqrt{\lambda}}}{N} \right)_{ij} \right] [N^{1/2} q^j_{\lambda,g} - q_{\lambda,g}(y_j^{(N)})/N^{1/2}]$$
$$= (O^1_{\lambda,g})_i + (O^2_{\lambda,g})_i$$

where

$$(O^1_{\lambda,g})_i = \frac{1}{N^{3/2}} \sum_{j=1}^{N} \tilde{G}_{i\sqrt{\lambda}}(y_i^{(N)} - y_j^{(N)})\, q_{\lambda,g}(y_j^{(N)})$$
$$- \frac{1}{N^{1/2}} (G_{i\sqrt{\lambda}} V q_{\lambda,g})(y_i^{(N)})$$

$$(O^2_{\lambda,g})_i = -\frac{\sqrt{\lambda}}{4\pi N^{3/2}}\, q_{\lambda,g}(y_i^{(N)})$$

By direct computation,

$$E\left(\sum_{i=1}^{N}(O^1_{\lambda,g})_i^2\right)=\frac{N-1}{N^2}(1, G^2_{i\sqrt{\lambda}}Vq^2_{\lambda,g})_{L^2_V} - \frac{N-2}{N^2}\|G_{i\sqrt{\lambda}}Vq_{\lambda,g}\|^2_{L^2_V} \tag{2.7}$$

$$E\left(\sum_{i=1}^{N}(O^2_{\lambda,g})_i^2\right)=\frac{\lambda}{16\pi^2N^2}\|q_{\lambda,g}\|^2_{L^2_V} \tag{2.8}$$

where

$$(G^2_{i\sqrt{\lambda}}f)(x)=\int_{\mathbf{R}^3}[G_{i\sqrt{\lambda}}(x-y)]^2 f(y)\,dy$$

From (2.7) and (2.8) we infer

$$\lim_{N\uparrow\infty} E\left(\sum_{i=1}^{N}(O^l_{\lambda,g})_i^2\right)=0, \qquad l=1,2$$

which together with the invertibility of the matrix

$$\left[\left(\alpha_i+\frac{\sqrt{\lambda}}{4\pi N}\right)\delta_{ij}-\left(\frac{\tilde{G}_{i\sqrt{\lambda}}}{N}\right)_{ij}\right]$$

gives us that for any $g\in L_2(\mathbf{R}^3)$ and λ sufficiently large

$$\lim_{N\uparrow\infty}\left\{\frac{1}{N}\sum_{i=1}^{N}[Nq^i_{\lambda,g}-q_{\lambda,g}(y_i^{(N)})]^2\right\}^{1/2}=0 \tag{2.9}$$

on a set of configurations $Y^{(N)}$ of measure going to 1 as N goes to ∞.

The proof of Theorem 1 follows now easily from

$$\begin{aligned}
&\left(f,\left[(-\Delta_{N\alpha^{(N)},Y^{(N)}}+\lambda)^{-1}-\left(-\Delta-\frac{V}{\alpha}+\lambda\right)^{-1}\right]g\right)\\
&=\sum_{i=1}^{N}q^i_{\lambda,g}(G_{i\sqrt{\lambda}}f)(y_i^{(N)})-\left(f,(-\Delta+\lambda)^{-1}\frac{V}{\alpha}\left(-\Delta-\frac{V}{\alpha}+\lambda\right)^{-1}g\right)\\
&=\sum_{i=1}^{N}[q^i_{\lambda,g}-q_{\lambda,g}(y_i^{(N)})/N](G_{i\sqrt{\lambda}}f)(y_i^{(N)})\\
&\quad+\frac{1}{N}\sum_{i=1}^{N}q_{\lambda,g}(y_i^{(N)})(G_{i\sqrt{\lambda}}f)(y_i^{(N)})-\int_{\mathbf{R}^3}(G_{i\sqrt{\lambda}}f)(x)\,q_{\lambda,g}(x)\,V(x)\,dx\\
&\leqslant\sup_x|G_{i\sqrt{\lambda}}f|(x)\left\{\frac{1}{N}\sum_{i=1}^{N}[Nq^i_{\lambda,g}-q_{\lambda,g}(y_i^{(N)})]^2\right\}^{1/2}\\
&\quad+|\eta^f_{\lambda,g}(Y^{(N)})-E[\eta^f_{\lambda,g}(Y^{(N)})]|
\end{aligned} \tag{2.10}$$

where

$$\eta^f_{\lambda,g}(Y^{(N)}) = \frac{1}{N}\sum_{i=1}^{N} q_{\lambda,g}(y_i^{(N)})(G_{i\sqrt{\lambda}}f)(y_i^{(N)})$$

Again by direct computation

$$\begin{aligned} E\,|\eta^f_{\lambda,g} - E\eta^f_{\lambda,g}|^2 &= \frac{1}{N}\left\{\int_{\mathbf{R}^3} (G_{i\sqrt{\lambda}}f)^2(x)\, q^2_{\lambda,g}(x)\, V(x)\,dx \right. \\ &\quad \left. - \left[\int_{\mathbf{R}^3} (G_{i\sqrt{\lambda}}f)(x)\, q_{\lambda,g}(x)\, V(x)\,dx\right]^2\right\} \\ &\leqslant \frac{C}{N}\sup_x |G_{i\sqrt{\lambda}}f|^2(x)[\|q_{\lambda,g}\|^2_{L^2_V} + (1, |q_{\lambda,g}|)^2_{L^2_V}] \end{aligned} \tag{2.11}$$

From (2.9)–(2.11), taking into account that $\sup_x |G_{i\sqrt{\lambda}}f|(x) \leqslant c\,\|f\|_2$ and our assumptions on V, we conclude that for every $g \in L_2$, on a set of $Y^{(N)}$ of measure going to 1 as N goes to infinity

$$\lim_{N\uparrow\infty} \{|(f, [(-\Delta_{N\alpha^{(N)}, Y^{(N)}} + \lambda)^{-1} - (-\Delta - V/\alpha + \lambda)^{-1}]\, g)|/\|f\|_2\} = 0$$

concluding the proof of Theorem 1. ∎

3. CONCLUSIONS

The Hamiltonian with zero-range potentials or point interactions supported on a discrete set of points $Y^{(N)}$ is defined by (1.1) in an implicit way, via its resolvent.

The law of large numbers guarantees the convergence of the measure

$$\frac{1}{N}\sum_{y_i^{(N)} \in Y^{(N)}} \delta(x - y_i^{(N)})$$

to $V(x)\,dx$ when N goes to infinity if $V(x)$ is the common distribution of the independently distributed vectors $y_i^{(N)} \in \mathbf{R}^3$. We proved that the same property holds for the Hamiltonian (1.1), together with a corresponding central limit theorem.

It is possible to analyze, in an analogous way, the convergence of the scattering data for the approximate Hamiltonians to the limit ones.

Notice that in the limit we studied the wave number is kept finite, the interparticle distance is of order $N^{-1/3}$, and the scattering parameter α^{-1} of order N^{-1}. The wavelength of the incoming particle "sees" only the density of scattering lengths $\alpha^{-1}(x)\ V(x)$.

Higher energy limits are very important in applications. In particular, the one where the wavelength is of the same order as the interparticle distance is the relevant case in many problems of scattering by liquids and solids. As we mentioned in the introduction, there does not seem to be a coherent approximation scheme available going beyond the linear term in the scattering parameter.

We conclude by mentioning that, with some additional technical complications, it is possible to extend the results given in this paper to potentials of the Rollnik class.

The Rollnik condition seems to be very natural in our context, being equivalent to the request that the system of "charges" $q_\lambda(x)$ has a finite energy. (For the same reason, we believe that the same results cannot be generalized to a larger class of potentials.)

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REFERENCES

1. H. Bethe and R. Peierls, Quantum Theory of the Diplon, *Proc. R. Soc. Lond.* **148A**:146 (1935).
2. E. Fermi, Sul moto dei neutroni nelle sostanze idrogenate (In Italian), *Ric. Sci.* **7**:13 (1936).
3. J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Springer-Verlag, New York, 1979).
4. L. H. Thomas, The Interaction Between a Neutron and a Proton and the Structure of H^3, *Phys. Rev.* **47**:903 (1935).
5. F. A. Berezin and L. D. Faddeev, A remark on Schrödinger's equation with a singular potential, *Sov. Math. Dokl.* **2**:372 (1961).
6. S. Albeverio, F. Gesztesy, R. Høgh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, Berlin, in press).
7. S. W. Lovesey, *Theory of Neutron Scattering from Condensed Matter*, Volume 1: *Nuclear Scattering*, Volume 2: *Polarization Effects and Magnetic Scattering* (Clarendon Press, Oxford, 1984).
8. B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms* (Princeton University Press, Princeton, New Jersey, 1971).
9. R. Figari, G. Papanicolaou, and J. Rubinstein, The point interaction approximation for diffusion in regions with many small holes, Preprint, Courant Institute of Mathematical Sciences (1986).
10. R. Figari, E. Orlandi, and A. Teta, The Laplacian in Regions with Many Small Obstacles: Fluctuations Around the Limit Operator, *J. Stat. Phys.* **41**:465 (1985).